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Bounding the distance from a reachable/controllable equisingular switched linear system to the set of non-reachable/uncontrollable ones

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Abstract. No necessary and sufficient condition for reachability of switched linear singular systems has been found in general, but only in the case of the so-called "equisingular systems". In this case, it is possible to obtain an upper bound for the distance between a controllable equisingular system and the set of uncontrollable ones.

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1. Introduction

Different bounds have been obtained for the distance of linear systems satisfying some qualitative properties to the set of those not satisfying them by D. Boley, W.S. Lu, J. Clotet, M.I. García-Planas, or R. Eising, for example (see [1], [2], [5], [6], [9], [10]). In [5] an upper bound for the distance between a reachable (controllable) switched linear system to the set of non-reachable (uncontrollable) ones is obtained, based on the algebraic characterization of reachability (controllability) of such systems provided by Z. Sun - S.S. Ge in [13].

In [4] a necessary and sufficient condition for reachability/controllability of "equisingular" switched linear systems is obtained. From this result, we deduce that reachability/controllability is a generic property and it makes sense to obtain bounds for the distance from a reachable/controllable system to the nearest non-reachable/uncontrollable one. This result may be useful when working with matrices with entries given with some parameter uncertainty.

The structure of the paper is as follows.

In Section §2, we summarize the definitions of switched linear systems, "equisingularity condition", and the results obtained by different authors related to controllability and reachability, including the algebraic characterization of reachability/controllability for "equisingular" linear systems.

In Section §3, we obtain and prove the value for the upper bound, from a reachable/controllable singular linear system to the set of non-reachable/uncontrollable ones.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbb{R})$ the set of matrices having n rows and m columns and entries in $\mathbb{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbb{R})$) and by $Gl_n(\mathbb{R})$ the group of non-singular matrices in $M_n(\mathbb{R})$.

2. Preliminaries

Switched linear systems consist of different subsystems of linear equations and a rule providing the switching between them.

The definitions below can be found in [10], [11], [12]. We include them here to make easier the reading of the paper.

A (non-singular) switched linear singular system is a system which consists of several linear subsystems, all of them regular, and a rule that determines the switching between them.

It can be written as

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the piecewise continuous input function, $y(t) \in \mathbb{R}^p$ is the output, $\sigma : [t_0, T) \longrightarrow M$, where t_0 is the initial time, $t_0 < T \leq \infty$, $M = \{1, \ldots, \ell\}$ is a right-continuous piecewise constant mapping (well-defined switching path) and for all $i \in M$, $A_i \in M_n(\mathbb{R}), B_i \in M_{n\times m}(\mathbb{R}), C_i \in M_{p\times n}(\mathbb{R})$.

A switched linear singular system is a system which consists of several linear subsystems, with at least one of them a singular system, and a rule that determines the switching between them.

It can be written as

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned}$$

where for all $i \in M$, $E_i, A_i \in M_n(\mathbb{R}), B_i \in M_{n\times m}(\mathbb{R}), C_i \in M_{p\times n}(\mathbb{R})$, and at least one of the matrices E_i is a singular matrix ($rk(E_i) < n$).

We will assume in this case that for all $i \in M$, the matrix pencil $\lambda_i E_i - A_i$ is regular and therefore there exists an unique solution for the system. Let

$\Phi(t, t_0, x_0, u, \sigma)$ be the solution $x(t)$ in the time t, with an initial condition $x(t_0) = x_0$.

For all singular linear subsystems we can consider a standard decomposition of the system from the Weierstraβ form of the matrix pencil $\lambda_i E_i - A_i$, $1 \leq i \leq \ell$. There exist $Q_i, P_i \in Gl_n(\mathbb{R})$, such that:

$$Q_i E_i P_i = \begin{pmatrix} I_{n_i} & 0 \\ 0 & N_i \end{pmatrix}, Q_i A_i P_i = \begin{pmatrix} G_i & 0 \\ 0 & I_{n-n_i} \end{pmatrix}, Q_i B_i = \begin{pmatrix} B_{i,1} \\ B_{i,2} \end{pmatrix}$$

where N_i are nilpotent matrices and G_i can be considered in Jordan reduced form. Let us denote by h the maximum of the nilpotent indices of matrices N_i.

Equisingular linear systems are those systems with $n_i = v$, $1 \leq i \leq \ell$.

Each of the subsystems of the system

$$\Sigma^C \qquad \begin{pmatrix} I_v & \\ & N_{\sigma(t)} \end{pmatrix} = \begin{pmatrix} G_{\sigma(t)} & \\ & I_{n-v} \end{pmatrix} x(t) + \begin{pmatrix} B^1_{\sigma(t)} \\ B^2_{\sigma(t)} \end{pmatrix} u(t)$$

splits into two independent equations (the so-called slow and fast parts):

$$\Sigma^S_{\sigma(t)} \qquad \dot{X}(t) = G_{\sigma(t)} X(t) + B^1_{\sigma(t)} u(t)$$

$$\Sigma^F_{\sigma(t)} \qquad N_{\sigma(t)} \dot{Y}(t) = Y(t) + B^2_{\sigma(t)} u(t)$$

Then we will call Σ^C_S the switched linear system having as subsystems the first ones above; that is to say,

$$\Sigma^S \qquad \dot{X}(t) = G_{\sigma(t)} X(t) + B^1_{\sigma(t)} u(t)$$

Before defining reachability and controllability, we recall the definition of admissible controls. Let us denote by $t_1, t_2, \dots, t_k$ the k switching discontinuous points in any given time interval $[\tau_1, \tau_2]$, $t_0 = \tau_1 < t_1 < t_2 < \dots < t_k < \tau_2$. That is to say, $\sigma(t) = \sigma(\tau_1)$ for $t \in [\tau_1, t_1)$, $\sigma(t) = \sigma(t_1)$ for $t \in [t_1, t_2), \dots$, $\sigma(t) = \sigma(t_k)$ for $t \in [t_k, \tau_2)$. Then the set of admissible controls in $[\tau_1, \tau_2]$ is the set:

$$U_\sigma([\tau_1, \tau_2]) = \left\{ u = [u^t_1, \dots, u^t_m] \right\}$$

with u_i, $1 \leq i \leq$ h-differentiable functions in the interval $[\tau_1, \tau_2]$ such that

$$\sum_{j=0}^{n_{\sigma(t_i)}-1} N^j_{\sigma(t_i)} B_{\sigma(t_i),2} u^{(j)}_{\sigma(t_i)}(t_i^+) = -(0 \; I_{n-n_i}) Q_i^{-1} \Phi(t_i^-, \tau_1, x_0, u, \sigma)$$

for $i \in \{0, 1, \dots, k\}$, with $t_0^- = \tau_1$, $C^h([\tau_1, \tau_2])$ the set of all h-differentiable functions in the interval $[\tau_1, \tau_2]$, $u^{(j)}_{\sigma(t_i)}(t_i^+)$ the j-derivative of $u_{\sigma(t_i)}(t)$ and $\Phi(t_i^-, \tau_1, x_0, u, \sigma)$ the left limit of $\Phi(t, \tau_1, x_0, u, \sigma)$ at $t = t_i$, $0 < \dots < i < \dots < k$, respectively.

Note that this set of admissible controls does not necessarily exist.

The system

$$\left\{ \begin{array}{rl} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{array} \right.$$

is said to be (completely) reachable if for any given initial time $t_0 \in \mathbb{R}$ and state $x_f \in \mathbb{R}^n$, there exists a real number $t_f > t_0$, a switching well-defined path $\sigma : [t_0, t_f] \longrightarrow M$ and an input $u : [t_0, t_f] \longrightarrow \mathbb{R}^m$, such that: $x_f = \Phi(t_f, t_0, 0, u, \sigma)$.

The system

$$\left\{ \begin{array}{rl} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{array} \right.$$

is said to be (completely) controllable if for any given initial time $t_0 \in \mathbb{R}$ and initial state $x_0 \in \mathbb{R}^n$, there exists a real number $t_f > t_0$, a switching well-defined path $\sigma : [t_0, t_f] \longrightarrow M$ and an input $u : [t_0, t_f] \longrightarrow \mathbb{R}^m$, such that: $x_f = \Phi(t_f, t_0, 0, u, \sigma)$.

We will summarize some of the results about reachability or controllability for singular switched linear systems below.

We will denote, given two matrices M, N, $< M, N > = \mathrm{Im}[N \mid MN \mid \ldots]$.

In 2009, in [4], the authors obtained a a necessary and sufficient condition for singular switched linear systems satisfying the "equisingularity condition" to be (completely) reachable/ (completely) controllable, not assuming the controls to be necessarily admissible. To state it more concretely, we need some further notation.

Let us denote by $V_1, \ldots, V_n$ the following vector subspaces:

$$V_1 = \sum_{i=1}^{m} Q_i\,(< G_i|B_{i,1} > \oplus < N_i|B_{i,2} >)$$

and, for $k > 1$,

$$V_k = \sum_{i=1}^{m} Q_i(< G_i|V_{k-1} > \oplus (< N_i|B_{i,2} >)$$

Clearly $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$. Moreover, if $V_j = V_{j+1}$ for some $j \in \{1, \ldots, n-1\}$, then $V_i = V_j$ for all $i > j$. In particular, $V_n = V_j$.

The algebraic characterization for reachability/controllability for "equisingular" switched linear systems is as follows.

Theorem 1 ([4]) *Let us assume that "equisingularity condition" holds (for all $i \in \{1, \ldots, \ell\}, n_i = v$). Then the following statements are equivalent.*

(a) Switched singular linear sytem Σ is (completely) reachable.
(b) Switched singular linear system Σ is (completely) controllable.
(c) $\bigcup_{i\in\{1,\dots,\ell\}} (V_N \oplus < N_i|B_{i,2} >) = \mathbb{R}^n$.

We can compare this result with the algebraic characterization of reachability/controllability for non-singular switched linear systems.

Theorem 2 ([13]) The following statements are equivalent.
(a) System $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ is (completely) reachable.
(b) System $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ is (completely) controllable.
(c) $\sum_{i_0,\dots,i_{n-1}\in\{1,\dots,m\}}^{j_1,\dots,j_{n-1}\in\{0,\dots,\ell-1\}} A_{i_{n-1}}^{j_{n-1}} \dots A_{i_1}^{j_1} \operatorname{Im} B_{i_0} = \mathbb{R}^n$.

When "equisingularity" condition does not hold, there are only necessary conditions and sufficient conditions, therefore no algebraic characterization. Concretely, the following results were obtained.

B. Meng-F.J. Zhang (2006) obtained a necessary condition and a sufficient condition for singular switched linear systems to be (completely) reachable, accepting only admissible controls.

Theorem 3 ([11]) Given any switched singular linear system Σ,

(a) if Σ is (completely) reachable, $V_n = \mathbb{R}^n$, and

(b) if $V_1 = \mathbb{R}^n$, $< N_i|B_{i,2} > = \mathbb{R}^{n-n_i}$ for all $i \in \{1,\dots,\ell\}$, Σ is (completely) reachable.

B. Meng- F.J. Zhang (2007) obtained a necessary condition and a sufficient condition for singular switched linear systems to be (completely) controllable, accepting only admissible controls.

Theorem 4 ([12]) Given any switched singular linear system Σ,

(a) if Σ is (completely) controllable, then $V_n = \mathbb{R}^n$, and

(b) if $V_n = \mathbb{R}^n$, $< N_i|B_{i,2} > = \mathbb{R}^{n-n_i}$ for all $i \in \{1,\dots,m\}$, then Σ is (completely) controllable.

Other results are those listed below.

Theorem 5 ([6]) Let us assume $m = 2$. That is to say, $M = \{1,2\}$. Let us assume that $V_1 = \mathbb{R}^n$ and there exists $i_0 \in M$ such that $< N_{i_0}|B_{i_0,2} > = \mathbb{R}^{n-n_{i_0}}$. Then the switched singular linear system Σ is (completely) reachable.

Note that, though the result above was stated for $m = 2$, it is obvious that it is also true for $m > 2$. Controls are not required to be admissible.

Theorem 6 ([7]) If the system Σ is (completely) reachable, then $V_n = \mathbb{R}^n$ and there exists $i_0 \in M$ such that $< N_{i_0} | B_{i_0,2} > = \mathbb{R}^{n-n_{i_0}}$.

Controls here are not required to be admissible. Even in the case where only admissible controls were considered, the condition in the statement is not a sufficient condition.

Remark 1 Note that the reachability/controllability properties do not depend on the signal σ. See [3] for further discussion about the admissible choices of sequence σ of switches and reachability.

3. Upper bound for the distance

Our aim is to find an upper bound for the distance from a singular switched linear system which is reachable/controllable to the nearest one which is not. We will denote by $d(\Sigma, U)$ to this distance. In all the Section the matrix norm considered will be the Frobenius norm. We recall that the Frobenius norm of a matrix $M = (m^i_j)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ is defined as

$$\|M\|_F = \sqrt{\sum_{1 \le i \le m} \sum_{1 \le j \le n} |m^i_j|^2}$$

and that the Frobenius norm satisfies the following inequality (submultiplicative property): for all $m \times n$-matrix M_1 and $n \times p$-matrix M_2, $\|M_1 M_2\| \le \|M_1\| \|M_2\|$.

We will consider $M = M_n(\mathbb{R})^{2\ell} \times M_{n\times m}(\mathbb{R})^{\ell}$ the set of matrices defining the state equations of the subsystems of an equisingular switched linear system.

Given two ordered sets of matrices in M, $((E_1, A_1, B_1), \dots, (E_\ell, A_\ell, B_\ell))$, $((X_1, Y_1, Z_1), \dots, (X_\ell, Y_\ell, Z_\ell))$ the distance between this two sets of matrices is defined as:

$$\|(E_1 - X_1, A_1 - Y_1, B_1 - Z_1, \dots, E_\ell - X_\ell, A_\ell - Y_\ell, B_\ell - Z_\ell)\|_F$$

Definition 1 The distance between the reachable/controllable system Σ,

$$\left\{\begin{array}{rl} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{array}\right.$$

and the nearest one which is non-reachable/ uncontrollable is defined as

$$\inf \|(\delta E_1, \delta A_1, \delta B_1), \dots (\delta E_\ell, \delta A_\ell, \delta B_\ell)\|_F$$

where $(\delta E_1, \delta A_1, \delta B_1), \dots (\delta E_\ell, \delta A_\ell, \delta B_\ell)$ is a a set of matrices such that $(E_1 + \delta E_1, A_1 + \delta A_1, B_1 + \delta B_1), \dots (E_\ell + \delta E_\ell, A_\ell + \delta A_\ell, B_\ell + \delta B_\ell)$ are the matrices defining the subsystems of a non-reachable/ uncontrollable switched linear system.

Remark 2 Distance $d(\Sigma, U) \leq d(\Sigma, U^E)$ where U^E is the set of switched "equisingular" linear systems which are non-reachable/ uncontrollable.

We will find an upper bound for $d(\Sigma, U^E)$.

Let us first introduce some further notation.

We will write

$$\overline{G}_i = \begin{pmatrix} G_i & 0 \\ 0 & 0 \end{pmatrix} \in M_n(\mathbb{R}), \quad i \in \{1, \dots, \ell\}$$

and denote by $M_S(\Sigma)$ the matrix having as columns the columns of matrix $(I_v \,|\, 0)V_n(\Sigma)$ and by $M_{F_i}(\Sigma)$ the matrices having as columns those of $< N_i | B_{i,2} >$.

Statement (c) in Theorem 1 can be written, using this notation, as:

(c') $\operatorname{rk} M_S(\Sigma) = v$ and there exists $i_0 \in \{1, \dots, \ell\}$ such that $\operatorname{rk} M_{F_{i_0}}(\Sigma) = n - v$.

Or equivalently, Σ is nonn-reachable if, and only if, Σ is uncontrollable if, and only if,

(c") $\operatorname{rk} M_S(\Sigma) < v$ or for all $i \in \{1, \dots, \ell\}$, $\operatorname{rk} M_{F_i}(\Sigma) < n - v$.

In order to find the upper bound, we first notice the following fact.

Lemma 1 $d(\Sigma, U^E) \leq d(\Sigma, \Sigma^C) + d(\Sigma^S, U_S)$ where U_S is the set of switched linear systems $\Sigma' : \dot{X}(t) = A_{\sigma(t)}X(t) + Bu(t)$ with $A \in M_v(\mathbb{R})$, $B \in M_{v \times m}(\mathbb{R})$ with $\operatorname{rk} M(\Sigma') < v$.

The proof straightforwardly follows from the fact that the $d(\Sigma^S, U_S)$ is equal to the distance from Σ^C to a switched linear system having the same fast parts than Σ^C but is non-reachable/ uncontrollable.

Lemma 2 Let us assume that $\dim < N_i | B_{i,2} > = n - v$ for all $i \in \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$. Then $d(\Sigma, U^E) \leq \sum_{i \in \{i_1, \dots, i_k\}} d(\Sigma_i^F, U_F)$, where U_F is the set of of non-reachable/ uncontrollable singular linear systems $E\dot{Y}(t) = Y(t) + Bu(t)$ with $E \in M_{n-\mu}(\mathbb{R})$, $B \in M_{(n-v)\times m}(\mathbb{R})$.

To prove this Lemma it suffices to take into account that this bound is the distance from system Σ^C to a switched linear system having fast parts for indexes $i_1, \dots, i_k$ replaced in such a way that now for all $i \in M = \{1, \dots, \ell\}$, $< N_i | B_{i,2} > < n - v$.

Finally, we can find an upper bound.

Theorem 7

$$d(\Sigma, U) \leq \min\{C_1, C_2\}$$

where

$$C_1 = \min_{\lambda_1,\dots,\lambda_\ell \in \mathbb{R}} \sigma_v(G_1 - \lambda_1 I_v | \dots | G_\ell - \lambda_\ell I_v | B_\ell^1)$$

and

$$C_2 = \min_{\substack{i \in \{1,\dots,\ell\} \text{ such that} \\ \operatorname{rk}[B_i^2 | N_i B_i^2 | \dots N_i^{h-1} B_i^2] = n - v}} \left(\min_{\lambda \in \mathbb{R}} \sigma_{n-v}(N_i - \lambda I_{n-v} | B_i^2) \right)$$

Proof.

We will use a result in [8] to determine that $d(\Sigma^S, U_S)$ is bounded by

$$\min_{\lambda_1,\dots,\lambda_\ell \in \mathbb{R}} \sigma_v(G_1 - \lambda_1 I_v | \dots | G_\ell - \lambda_\ell I_v | B_\ell^1)$$

On the other hand, notice that for all $i \in \{i_1, \dots, i_k\}$ such that $\dim < N_i | B_{i,2} > = \operatorname{rk}[B_i^2 | N_i B_i^2 | \dots | N_i^{h-1} B_i^2] = n - v$, we can consider the system $S_i \quad \dot{x}(t) = N_i x(t) + B_i^2 u(t)$, which is reachable/controllable.

In [8], an upper bound for the distance of each system $S_i : \quad \dot{x}(t) = N_i x(t) + B_i^2 u(t)$ to the nearest non-reachable/uncontrollable system (that is to say, $\dim < N_i | B_{i,2} > < n - v$) is obtained:

$$\min_{\lambda \in \mathbb{R}} \sigma_{n-v}(N_i - \lambda I_{n-v} | B_i^2)$$

4. Conclusions

In this paper an upper bound for the distance between a reachable/controllable equisingular switched linear system is provided. Future works might allow to obtain a lower bound.

References

[1] D. Boley, "Estimating the Sensitivity of the Algebraic Structure of Pencils with Simple Eigenvalue Estimates", SIAM J. Matrix Anal. Appl, 11, no. 4, pp. 632-643 (1990).

[2] D. Boley and W.-S. Lu, "Measuring how far a controllable system is from an uncontrollable one", IEEE Trans. on Automatic Control, AC-31, pp. 249-251 (1986).

[3] M.V. Carriegos, H. Díez-Machío, A. Prieto-Castro, L. Rivas-Morán, "Switched linear systems: A study using local automata", Proceeding of the International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE (2008).

[4] J. Clotet, J. Ferrer and M.D. Magret, "Switched singular linear systems", Proceedings of the 17th Mediterranean Conference on Control and Automation. Thessaloniki, Greece, pp. 1343-1347 (2009).

[5] J. Clotet and M.D. Magret, "Switched singular linear systems and reachability". Proceedings of PHYSCON 2011. Leon, Espaa (2011).

[6] J. Clotet and M.D. Magret, "Sistemas lineales conmutados singulares. Accesibilidad". Proceedings of XXIII CEDYA / XIII CMA 2013: SEMA SIMAI Springer Series, pp. 1047-1054 (2013).

[7] J. Clotet and M.D. Magret, "Upper Bounds for the Distance between a Controllable Switched Linear System and the Set of Uncontrollable Ones". Mathematical Problems in Engineering 2013, pp. 1-9 (2013).

[8] J. Clotet, Mª I. García-Planas and M.D. Magret, "Estimating distances from quadruples satisfying stability properties to quadruples not satisfying them", Linear Algebra and its Applications, 332-334, pp. 541-567 (2001).

[9] R. Eising, "Between controllable and uncontrollable", Syst. Control Letters 4 no. 5, pp. 263-264 (1984).

[10] R. Eising, "The distance between a system and the set of uncontrollable systems", Memo COSOR 82-19, Eindhoven Univ. Technol., The Netherlands (1982).

[11] B. Meng and F.J. Zhang, "Reachability conditions for switched linear singular systems", IEEE Transactions on Automatic Control, 51 (2006).

[12] B. Meng and F.J. Zhang, "Controllability conditions for switched linear singular systems ", Dynamics of continuous, discrete and impulsive systems. Series B, 14 (2007).

[13] Z. Sun and S. S. Ge, Switched Linear Systems, Springer Verlag, London (2005).